Duality Symmetry of $\mathcal{N}\!=\!4$ Yang-Mills Theory on T^3

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Abstract

We study the spectrum of BPS states in $\mathcal{N}\!=\!4$ supersymmetric $U(N)$ Yang-Mills theory. This theory has been proposed to describe M-theory on T^3 in the discrete light-cone formalism. We find that the degeneracy of irreducible BPS bound states in this model exhibits a (partially hidden) $SL(5, \mathbf{Z})$ duality symmetry. Besides the electro-magnetic symmetry, this duality group also contains Nahm-like transformations that interchange the rank N of the gauge group with some of the magnetic or electric fluxes. In the M-theory interpretation, this mapping amounts to a reflection that interchanges the longitudinal direction with one of the transverse directions.

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1. Introduction

In this paper we will consider $\mathcal{N}=4$ supersymmetric Yang-Mills theory in 3+1 dimensions. This model is described by the Lagrangian

$$
S = -\frac{1}{g^2} \int d^4x \, \mathrm{tr}\Big(\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(D_\mu X^I)^2 + \psi\Gamma^i D_i \psi
$$

$$
+ \psi\Gamma^I[X^I, \psi] + \frac{1}{4}[X^I, X^J]^2\Big) \tag{1.1}
$$

with $I = 1,\ldots,6$, and can be thought of as the dimensional reduction of $\mathcal{N}=1$ SYM theory from ten to four dimensions. Elegant semi-classical studies have revealed that the spectrum of dyonic BPS-saturated states in this theory exhibits an exact symmetry between electric and magnetic charges [1, 2, 3]. This S-duality symmetry is expected to extend into the full quantum regime, thereby providing an exact mapping between the strong coupling and weak coupling regions. Although the recent breakthroughs in non-perturbative supersymmetric gauge theory and string theory have produced a substantial amount of evidence for this duality conjecture, finding an explicit construction of this duality mapping still seems as difficult as ever.

The S-duality of the $\mathcal{N}=4$ model forms an important ingredient in the matrix theory formulation of 11-dimensional M-theory [4]. Matrix theory proposes a concrete identification between (the large N limit of) the $U(N)$ SYM-model defined on a three-torus T^3 and type IIA or B superstring theory compactified on a two-torus T^2. This correspondence with perturbative string theory becomes most precise in the limit when one of the directions of the three torus becomes much larger than the others. In this IR limit one approaches an effective conformal field theory, that was argued in [7, 8, 9] to correspond to a second quantized IIA string theory. A particularly striking consequence of this conjectured correspondence is that the S-duality of the gauge model gets mapped to a simple T-duality in the string theory language [5, 6].

Central in the correspondence with M-theory on T^3 is the following construction of the 11-dimensional supersymmetry algebra in terms of the SYM degrees of freedom [10]. The generators of the four-dimensional $\mathcal{N}=4$ supersymmetry algebra can be conveniently combined into one single $SO(9,1)$ spinor supercharge Q_α, by considering the 4D SYM model as the dimensional reduction of ten-dimensional $\mathcal{N}=1$ SYM theory. In this ten dimensional notation the fermionic fields transform under the supersymmetry according to $\delta\psi = \Gamma^{\mu\nu}F_{\mu\nu}\epsilon$, and the corresponding supercharge Q_α is equal to

$$
Q = \int_{T^3} \mathrm{tr}\left[\Gamma^r \Psi E_r - \Gamma^0\Gamma^{rs}\Psi\frac{1}{2}F_{rs}\right]. \tag{1.2}
$$

Here and in the following the indices r, s run from 1 to 9. As the spatial part of our space-time manifold is compact we have an additional global supersymmetry: the action

is invariant under adding a constant spinor to ψ via $\delta\psi = \tilde{\epsilon}$. We denote the corresponding supercharge by $\tilde{Q}$

$$\tilde{Q} = \int_{T^3} \text{tr}\,\Psi, \tag{1.3}$$

where we put the volume of the three torus (and all its sides) equal to one. The supersymmetry algebra is

$$\begin{aligned}
\left\{\tilde{Q}_\alpha, \tilde{Q}_\beta\right\} &= N\delta_{\alpha\beta} \\
\left\{Q_\alpha, \tilde{Q}_\beta\right\} &= Z_{\alpha\beta} \\
\left\{\bar{Q}_\alpha, Q_\beta\right\} &= 2\Gamma^0 H + 2\Gamma^i P_i
\end{aligned} \tag{1.4}$$

where the central charge term

$$Z = \Gamma^0 \Gamma^i e_i - \Gamma^{ij} m_{ij} \tag{1.5}$$

consists of the total electric and magnetic flux through the three-torus, defined via

$$\begin{aligned}
e_i &= \int_{T^3} \text{tr}\,E_i, \\
m_{ij} &= \int_{T^3} \text{tr}\,F_{ij}.
\end{aligned} \tag{1.6}$$

H in (1.4) denotes the SYM Hamiltonian and the quantities P_i are the integrated energy momentum fluxes, defined (in 3+1 notation) via

$$P_i = \int_{T^3} \text{tr}(E_j F_{ji} + \Pi^I D_i X_I + \frac{1}{2} i\Psi^{\text{T}} D_i \Psi), \tag{1.7}$$

where Π_I defines the conjugate momentum to the Higgs scalar field X_I. In writing the above supersymmetry algebra we have assumed that the $U(1)$ zero mode part of Π_I vanishes. From the eleven dimensional M-theory perspective, this means that we assume to be in the rest-frame in the uncompactified space directions.

In the next table we summarize the interpretation of the various fluxes of $3+1$-dimensional SYM theory in terms of charges of 11-dimensional M-theory on T^3 and 10-dimensional IIA string theory compactified on T^2. The latter correspondence, with the IIA string, proceeds by writing $T^3 = T^2 \times S^1$ and interpreting the third direction (along the S^1) as the eleventh direction of M-theory. This leads to the following list of

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correspondences (here i, j run from 1 to 2)

N	H
e_3	e_i
m_3	m_i
p_3	p_i

$\leftrightarrow$

p_+	p_-
p_3	p_i
m_{ij}	m_{3j}
m_{3+}	m_{i+}

$\leftrightarrow$

p_+	p_-
q_0	p_i
m_{ij}	w_j
w_+	m_{i+}

  4D SYM   M-theory on T^3 IIA String on T^2

Here q_0 denotes the D-particle number in the IIA string theory. In the second and third table, the integers m_{ij} – corresponding to the SYM magnetic fluxes – denote the wrapping numbers of the M-theory membrane around the T^3 and the $D2$-brane around the T^2 respectively. The M-theory membranes wrapped m_{i3} times around the 3-direction turn into IIA strings with winding number w_i.

States with non-zero momentum flux $P_i = p_i$ in the SYM theory correspond to membranes that are wrapped around the longitudinal +-direction. In the original proposal of [4], M-theory arises in the limit $N \to \infty$ while restricting the spectrum of the $\mathcal{N} = 4$ SYM model to the subspace of states that have energy of order $1/N$. This limit amounts to a decompactification of the longitudinal +-direction. States with non-zero p_i will thus correspond to infinite energy configurations containing membranes stretched along the light-cone direction. All finite energy states therefore must have $p_i = 0$.

An interesting proposal made in [11] extends the matrix theory conjecture to finite N, via a discrete light-cone formalism. In this set-up, one compactifies the light-direction on a circle and considers the truncation of the M-theory Hilbert space to a given finite, discrete light-cone momentum identified with the rank N. In this case, the longitudinal membranes are no longer infinitely massive, and must be naturally included in the spectrum.

The correspondence with string theory and M-theory gives a number of predictions concerning the duality properties of the gauge theory. These predictions in particular concern the BPS spectrum of the SYM model at finite N as well as the behaviour of the model at large N. For example, from the above table it is seen immediately that electric-magnetic duality in the SYM theory indeed must follow from the T-duality on the two-torus, as first noted in [5]. The T-duality that exchanges the D-particles with the D2-branes and the KK momenta with the NS winding numbers in type IIA string theory, when translated to the first table indeed gives rise to the S-duality that interchanges all the electric and magnetic fluxes. Via this correspondence, the complete U-duality symmetry $SL(2, \mathbf{Z}) \times SL(3, \mathbf{Z})$ of M-theory on T^3 is expected to be realized as an exact symmetry of the matrix formalism [5, 6].

For finite N, however, there are reasons to suspect that the duality group that acts on the BPS sector is in fact enlarged. Following the proposal of [11], it should describe

M-theory with more than 3 directions compactified, and correspondingly one no longer needs to restrict to states with vanishing momentum flux p_i. BPS states at finite N can therefore carry a total of 10 charges, labeled by (N, e_i, m_i, p_i). The goal of our study is to determine the explicit form of the BPS degeneracies for finite N as a function of these charges, and thereby exhibit its full duality symmetry. We will approach this problem in two ways: first from M-theory and then directly from the $\mathcal{N} = 4$ SYM model. Our main finding is that the spectral degeneracy of individual bound states from both points of view is indeed identical, and furthermore exhibits a full $SL(5, \mathbf{Z})$ duality symmetry. Part of this large duality group also acts on the rank N of the gauge group. We will comment on the geometric origin of this symmetry in the final section.

2. BPS SPECTRUM FROM M-THEORY

The BPS-states that we will consider respect $1/4$ of all supersymmetries. Every such BPS-state in a fixed multiplet satisfies

$$\left(\epsilon^\alpha Q_\alpha + \tilde{\epsilon}^\alpha \tilde{Q}_\alpha \right) |\text{BPS}\rangle = 0 \tag{2.1}$$

for certain fixed collection of $SO(9, 1)$ spinors ϵ and $\tilde{\epsilon}$. These spinors are, up to an overall factor, completely determined by the set of charges.

By taking the commutator with $\bar{Q}$ and $\tilde{Q}$ in the preceding equation we get two conditions for the spinors ϵ and $\tilde{\epsilon}$

$$2\Gamma^0 H \epsilon + 2\Gamma^i P_i \epsilon + \Gamma^0 Z \tilde{\epsilon} = 0,$$

$$N\tilde{\epsilon} + Z^\dagger \epsilon = 0. \tag{2.2}$$

Plugging $\tilde{\epsilon}$ in the first equation gives the following equation for ϵ

$$\left(\Gamma^0 H' + \Gamma^i P_i' \right) \epsilon = 0 \tag{2.3}$$

where H' and P_i' denote the Hamiltonian and momentum fluxes with the zero-mode contributions removed. Explicitly,

$$H = \frac{1}{2N}(e_i^2 + m_i^2) + H'$$

$$P_i = (e \wedge m)_i / N + P_i'. \tag{2.4}$$

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From the last equation, we see that the eigenvalues of P'_i are equal to $p'_i = \kappa_i/N$ with

$$\kappa_i = Np_i - (e \wedge m)_i \qquad (2.5)$$

The operator in equation (2.3) should have eigenvalues equal to zero. This is only the case when the magnitude of H' and the length of P'_i are equal, $H' = |P'_i|$. Using this relation, we can write equation (2.3) in the following way

$$\left(\Gamma_0 + \Gamma^i \hat{\kappa}_i\right) \epsilon = 0 \qquad (2.6)$$

where the vector $\hat{\kappa}_i$ denotes the unit vector in the direction of κ_i.

Now we are ready to determine the detailed BPS spectrum from discrete light-cone M-theory. To this end it is useful to introduce the notion of an *irreducible* BPS state, as a state that contains only one *single* BPS bound state. Indeed, general BPS states can in principle combine more than one such bound state into one second quantized BPS state. The combined state will still be BPS, provided each of the irreducible constituent bound states all left invariant by the *same* set of supersymmetries, *i.e.* with the same ϵ and $\tilde{\epsilon}$.

The degeneracy of irreducible BPS states is determined almost uniquely from U-duality invariance, and from the known BPS spectrum of perturbative string states. The relevant U-duality group for our case turns out to be as large as the complete $SL(5, \mathbf{Z})$ duality group of string theory on T^3, *i.e.* M-theory on T^4. At first sight, the appearance of this large symmetry may seem surprising, in particular since the supersymmetry algebra (1.4) itself is not (manifestly) $SL(5, \mathbf{Z})$ invariant. The origin of this large duality group can be understood, however, by noting that the discrete light-cone version of M-theory on T^3 can be obtained from M-theory on T^4 via a suitable limit, essentially by applying an infinite boost to one of the compactified directions of the T^4. The BPS spectrum should be invariant under such boosts.[1]

The 10 quantum numbers (N, e_i, m_i, p_i) can be combined into a 5×5 anti-symmetric matrix on which $SL(5, \mathbf{Z})$ acts by simultaneous left- and right-multiplication. The following bilinear combinations

$$K_i = (Np_i - (e \wedge m)_i, \; p \cdot m, \; p \cdot e) \qquad (2.7)$$

[1] Alternatively, we can interpret the $U(N)$ SYM model as the low energy description of all possible bound states of N D3 branes of IIB string theory on T^3. In this correspondence, e_i denotes the NS string and m_i the D-string winding number, while p_i is the KK momentum. The SYM theory should thus encompass all BPS bound states of this system. It should be emphasized, however, that from this reasoning we should only expect the BPS *degeneracy* formula to be $SL(5, \mathbf{Z})$ symmetric, while of course the energy spectrum is not, since the Yang-Mills model only represents a particular limit of the N D3 brane system.

transform as 5 vector under this duality. From this we deduce that the unique $SL(5, \mathbf{Z})$ invariant scalar combination we can make out of the ten charges is the integral length of this five-vector

$$|K| = \gcd(Np_i - (e \wedge m)_i,\, p{\cdot}m\,,\, p{\cdot}e\,) \tag{2.8}$$

Duality invariance thus predicts that the degeneracy of irreducible BPS bound states should be expressible in terms of this quantity $|K|$ only.

To determine the explicit degeneracy formula, we note that by using the U-duality symmetry, any irreducible bound state can be rotated into a state that carries only KK momentum and NS string winding. Such a state must necessarily be made up from a single fundamental IIA string. The invariant $|K|$ for this perturbative string state simply reduces to the bilinear combination of momenta and winding numbers that determines (via the BPS restriction) the oscillator level of the string. The number of irreducible BPS bound states is therefore counted by means of the chiral string partition function

$$\sum_K c(K)q^K = (16)^2 \prod_n \left(\frac{1 + q^n}{1 - q^n} \right)^8 \tag{2.9}$$

with $K = |K|$ as defined in (2.8).

General BPS states of discrete light-cone gauge M-theory may consist of more than one irreducible bound state. the total BPS condition requires that the charges of these separate bound states must be compatible. The complete second quantized partition sum is obtained by taking into account all possible such ways of combining individual bound states into a second quantized configuration with a given total charge. More detailed comments on the combinatorial structure of the second quantized BPS partition sum will be given in the next section.

3. BPS-SPECTRUM OF $\mathcal{N}{=}4$ SYM ON T^3

The above description of the BPS spectrum can be reproduced directly from the $U(N)$ gauge theory on T^3 as follows. First let us recall the definition of the electro-magnetic flux quantum numbers. To this end it is useful to decompose the $U(N)$ gauge field into a trace and a traceless part

$$A_\mu = A_\mu^{U(1)}\mathbf{1} + A_\mu^{SU(N)} \tag{3.1}$$

and allow the fields on T^3 to be periodic up to gauge transformations of the form

$$A_i^{SU(N)}(x + a_j) = \Omega_j A_i^{SU(N)}(x)\Omega_j^{-1}$$

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$$A_i^{U(1)}(x + a_j) = A_i^{U(1)}(x) - 2\pi m_{ij}/N \tag{3.2}$$

with m_{ij} integer. Here a_i with $i = 1, 2, 3$ denote the translation vectors that define the three torus T^3.[2] The $SU(N)$ rotations Ω_i must satisfy the $\mathbf{Z}_N$ cocycle conditions

$$\Omega_j \Omega_k = \Omega_k \Omega_j e^{2\pi i \mu_{jk}/N} \tag{3.3}$$

for integer μ_{jk}. The quantities μ_{ij}/N define the 't Hooft Z_N magnetic fluxes [12] and m_{ij}/N can be identified with the $U(1)$ magnetic flux defined in (1.6). The integers μ_{ij} and m_{ij} are restricted via the condition that the combined gauge transformation (3.2) defines a proper $U(N)$ rotation. This requirement translates into the Dirac quantization condition that the total flux $(\mu_{ij} - m_{ij})/N$ must be an integer. Hence $m_{ij} = \mu_{ij}(\text{mod } N)$.

Electric flux carried by a given state is defined via the action of quasi-periodic gauge rotations $\Omega[\mathbf{n}]$ defined via

$$\Omega_j \Omega[\mathbf{n}] = \Omega[\mathbf{n}] \Omega_j e^{2\pi i n_j/N} \tag{3.4}$$

with integer n_j. Such gauge rotations will preserve the boundary condition (3.2) on the gauge fields. A state $|\psi\rangle$ is defined to carry $SU(N)$ flux ϵ_j if it satisfies the eigenvalue condition

$$\widehat{\Omega}[\mathbf{n}]|\psi\rangle = e^{\frac{2\pi i}{N} n^j \epsilon_j} |\psi\rangle \tag{3.5}$$

where $\widehat{\Omega}[\mathbf{n}]$ denotes the quantum operator that implements the gauge rotation $\Omega[\mathbf{n}]$ on the state $|\psi\rangle$. Similarly as for the magnetic flux, the electric flux receives an overall $U(1)$ contribution e_i defined in (1.6). The abelian and non-abelian parts of the flux must again be related via $e_i = \epsilon_i(\text{mod } N)$.

To determine the supersymmetric spectrum for a given set of charges, the idea is to first reduce the phase space of the $U(N)$ SYM model to the space of classical supersymmetric configurations, and then to quantize this BPS reduced phase space. The justification for this procedure should come from the high degree of supersymmetry in the problem, while furthermore the degeneracy of BPS states is known to be a very robust quantity.

To obtain the reduced phase space, we recall that the SUSY transformation for the fermionic partners of the Yang-Mills fields reads (again using ten-dimensional notation)

$$\delta\Psi = \left(E_r \Gamma^{0r} + \frac{1}{2} F_{rs} \Gamma^{rs}\right) \epsilon + \tilde{\epsilon} \tag{3.6}$$

The BPS restriction requires that the right-hand side vanishes for those ϵ and $\tilde{\epsilon}$ determined in the previous section. Thus in particular we can use the equation (2.2) to express $\tilde{\epsilon}$ in

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[2]We repeat here that for simplicity we mostly take T^3 to be cubic with sides of length 1.

terms of ϵ and the $U(1)$ zero modes. The result is

$$\delta\Psi = \left(E'_r \Gamma^{0r} + \frac{1}{2} F'_{rs} \Gamma^{rs} \right) \epsilon = 0. \tag{3.7}$$

where the primed quantities are equal to the un-primed ones with the constant $U(1)$ parts removed. For BPS-states in a fixed multiplet, supersymmetry is unbroken for ϵ satisfying the equation (2.3) above. Hence for these BPS-states the following must hold for all spinors

$$\left(E'_r \Gamma^{0r} + \frac{1}{2} F'_{rs} \Gamma^{rs} \right) \left(\Gamma^0 - \Gamma^k \hat{\kappa}_k \right) \epsilon = 0. \tag{3.8}$$

Note that $\Gamma^0 - \Gamma^k \hat{\kappa}_k$ acts like a projection operator on the space of spinors satisfying equation (2.3). We conclude that the matrix in spinor space in the last equation has to vanish. This is the case when E' and F' satisfy the following two conditions

$$E'_i \hat{\kappa}^i = 0$$

$$E'_{[r} \hat{\kappa}_{s]} = F'_{rs} \tag{3.9}$$

From now one we shall omit the prime, and simply denote by E and F_{ij} the $U(N)$ fields without the $U(1)$ constant mode. We will also return to a 3+1-dimensional notation, and for additional notational convenience, use the $SL(3, \mathbf{Z})$ symmetry to rotate the three-vector κ_i defined in (2.5) in the 3 direction. So we will choose coordinates such that

$$\kappa_3 = N p_3 - e_i m_{i3} \tag{3.10}$$

$$\kappa_j = N p_j - e_i m_{ij} - e_3 m_{3j} = 0 \qquad i, j = 1, 2.$$

Here and from now on the indices i, j run from 1 to 2. From the second condition in (3.9) we read off that the gauge and Higgs fields are flat on the plane perpendicular to $\hat{\kappa}$, meaning

$$F_{ij} = 0,$$

$$D_i X_J = 0, \tag{3.11}$$

$$[X_I, X_J] = 0.$$

In addition we have

$$E_i = F_{3i}$$

$$\Pi_I = D_3 X_I. \tag{3.12}$$

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Finally, the first condition in (3.9) simply becomes

$$E_3 = 0. \tag{3.13}$$

We can interpret this constraint equation as a gauge invariance condition under arbitrary local shifts in the longitudinal gauge field A_3. We can therefore exploit this gauge invariance by putting $A_3 = 0$. The relations (3.11) then simplify to the statement that the transversal gauge fields A_i and Higgs scalars X_I satisfy the chiral 2D free field equations

$$\partial_0 A_i = \partial_3 A_i$$

$$\partial_0 X_I = \partial_3 X_I. \tag{3.14}$$

Thus we conclude that the BPS reduced theory is described by the left-moving chiral sector of a two-dimensional sigma model with target space given by the space of solutions to the flatness conditions (3.11), subject to the twisted boundary conditions specified by the electro-magnetic flux quantum numbers.

To determine the detailed properties of this sigma model, let us first consider the case with all electro-magnetic fluxes equal to zero. The only non-zero quantum numbers are therefore N and p_3. In this case we can parameterize the space of solutions to (3.11) by means of the orbifold sigma model on the N-fold symmetric product space

$$\frac{(\mathbf{R}^6 \times T^2)^N}{S_N} \tag{3.15}$$

where S_N denotes the permutation group of N elements, acting on the N copies of the transversal space $T^2 \times \mathbf{R}^6$.[3] To see that this is the right space, we observe that for solutions to (3.11) one can always choose a gauge in which all $U(N)$ valued fields take the form of diagonal matrices. Each such matrix field thus combines N separate scalar fields, corresponding to the N eigenvalues. The S_N permutation symmetry arises as a remnant of $U(N)$ gauge invariance, acting via its Weyl subgroup on the space of diagonal matrices. Finally, the flat transversal gauge fields A_i with $i = 1, 2$ give rise to *periodic* 2D scalar fields, since constant shifts in A_i by multiples of 2π are pure gauge rotations.

The model thus reduces to the free limit of type IIA matrix string theory [7, 8, 9] in the discrete light-cone gauge [11]. As explained in [9], the Hilbert space of the model decomposes into twisted sectors labeled by the partitions of N, in which the eigenvalue fields combine into a collection of 'long strings' of individual length n_k such that the total

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[3]This orbifold sigma model was first considered in relation with $\mathcal{N}=4$ SYM theory on the three torus in [13].

length adds up to $\sum_k n_k = N$. Each such string is made up from, say, n_k eigenvalues that, by their periodicity condition around the 3-direction are connected via a cyclic permutation of order n_k. In the M-theory interpretation, all these separate strings will indeed correspond to separate bound states, *i.e.* particles that each can move independently in the uncompactified space directions. The general form of the BPS partition function of symmetric product sigma models of the form (3.15) has been described in detail in [14].

In the following we will mainly concentrate on the irreducible states, describing one single BPS particle. These necessarily consist of one single string of maximal length. In the present case, with zero total electro-magnetic flux, this maximal string has total winding number N around the 3-direction. Correspondingly, its oscillation modes have energies that are quantized in units of $1/N$. Thus the degeneracy of states as a function of N and p_3 is obtained by evaluating the chiral superstring partition function, as given in (2.9), at oscillator level Np_3.

Next let us turn on the magnetic flux m_3. The space of solutions to (3.11) with twisted boundary conditions (3.2) around the transverse torus again takes the form of a symmetric product

$$\frac{(\mathbf{R}^6 \times T^2)^{N'}}{S_{N'}} \tag{3.16}$$

but where now $N' = \gcd(N, m_3)$. In order to visualize this reduction, we note that gauge rotations Ω_i with $\Omega_1 \Omega_2 = \Omega_2 \Omega_1 \exp(2\pi i m_3 / N)$, that define the twisted boundary conditions, can be chosen to lie within an $SU(k)$ subfactor of $U(N)$ where $k = N/\gcd(N, m_3)$. By decomposing the matrix valued fields according to the action of $SU(k) \otimes U(N')$ with $N' = \gcd(N, m_3)$, we can thus factor out a sector of $U(N')$ valued field variables that are unaffected by the twisted boundary conditions. Now following the same reasoning as before, these fields parameterize the symmetric product space of the above form. Note that in the particular case that $m_3 = 1$, the whole $SU(N)$ part of the moduli space of solution to (3.11) collapses to a point, so that only the $U(1)$ part survives.

By a very similar reasoning we can also include the electric flux e_3 in our description. Like with the magnetic flux, an electric flux $e_3 = 1$ has the effect of reducing the $SU(N)$ part of the vacuum moduli space (3.11) to a point, or rather, it projects out just one single supersymmetric state in the $SU(N)$ sector [15]. More generally, however, it can be seen that one can again factor out a $U(N')$ subfactor of the model, that is unaffected by both the electric and magnetic flux e_3 and m_3, where now

$$N' = \gcd(N, m_3, e_3). \tag{3.17}$$

The BPS sector for non-zero m_3 and e_3 is thus obtained by quantizing the supersymmetric orbifold sigma model on (3.16), with N' equal to (3.17).

The spectrum of irreducible BPS bound states is obtained as before, by considering the Hilbert space sector defined by the eigenvalue string of maximal length. The maximal winding number is now equal to N'. The total momentum along this string is determined by the remaining quantum numbers of the BPS state, and should be equal to

$$p_3 - e_i m_{i3}/N, \tag{3.18}$$

which is the total momentum of the BPS state minus the contribution from the $U(1)$ electro-magnetic fluxes. Notice that the latter contribution is in general fractional. However, since the oscillation modes of the long string states also have fractional oscillation number quantized in units of $1/N'$, this fractional total momentum (3.18) can in fact be obtained via integer string oscillation levels. The total oscillation level corresponding to this momentum flux is

$$K = N' \times (p_3 - e_i m_{i3}/N) \tag{3.19}$$

and it is easy to check using (3.10) that this is an integer. In fact, after taking the direction of κ_i again arbitrary, we find that the integer quantity (3.19) becomes equal to the $SL(5, \mathbf{Z})$ invariant length $|K|$ defined in (2.8) of the five-vector (2.7). In this way we reproduce the description of the BPS spectrum given in the previous section. In particular we find confirmation that the degeneracy of supersymmetric bound states has a discrete $SL(5, \mathbf{Z})$ duality invariance.

As mentioned already, the complete supersymmetric spectrum of the $\mathcal{N} = 4$ SYM model contains many more sectors, that in M-theory describe configurations of multiple BPS bound states. These correspond to the other twisted sectors in the orbifold model (3.16), describing multiple strings with separate lengths n_k with $\sum_k n_k = N'$. Via their zero modes, these strings each separately carry all the possible flux quantum numbers. The degeneracy of these separate states as a function of these charges is identical to the $SL(5, \mathbf{Z})$ invariant result just described. However, it turns out that the combinatorics by which many such states can be combined into one second quantized BPS state no longer respects the full $SL(5, \mathbf{Z})$ symmetry. For this it would be necessary that, via the compatibility of the individual BPS conditions, the 10 dimensional charge vectors of all constituent states must align in the same direction. It can be seen, however, that this alignment is not entirely implied: in the above notation, all charges must indeed align, *except* for the individual p_3 momenta. It appears therefore that the full second quantized BPS partition sum exhibits a somewhat smaller duality symmetry than $SL(5, \mathbf{Z})$. The symmetry that gets preserved, however, is still substantially bigger than the $SL(2, \mathbf{Z}) \times SL(3, Z)$ U-duality of M-theory on T^3. In the next section we will describe some of these extra symmetries in more detail.

4. Nahm Duality

An especially interesting class of duality transformations that act on BPS sectors of $U(N)$ SYM theory are those that interchange the rank N of the gauge group with the magnetic flux. These type of transformations are very similar to the Nahm-type transformations, considered e.g. in [16]. In the notation used above (with $\hat{\kappa}$ rotated in the 3-direction) the BPS reduced quantum phase space exhibits a manifest symmetry under the interchange

$$N \leftrightarrow m_3 \qquad m_i \rightarrow \epsilon^{ij} m_j \qquad e_i \leftrightarrow p_i, \tag{4.1}$$

as well as under its electromagnetic dual counterpart

$$N \leftrightarrow e_3 \qquad e_i \rightarrow \epsilon^{ij} e_j \qquad m_i \leftrightarrow p_i. \tag{4.2}$$

In particular one can verify that the relations (3.10) as well as the integer N' are invariant under these two mappings. This last type of duality symmetry is particularly interesting, because from M-theory, the symmetry (4.2) must be some manifestation of 11-dimensional covariance. Because of this possible relevance, it is of interest to know whether the symmetry perhaps extends to the full $\mathcal{N}=4$ model. Although we do not directly expect that this is the case, we cannot resist here to review the geometrical origin of this duality mapping, which in fact can be defined for arbitrary non-BPS gauge configurations.

Consider an arbitrary $U(N)$ gauge field $A = A_x + iA_y$ on T^2 with magnetic flux M. This configuration can be related to a dual $U(M)$ gauge field with flux N defined on the dual torus $\hat{T}^2$ as follows. The key idea of the construction is to consider the parameter family of connections on T^2 of the form (here a, b denote the $U(N)$ color indices)

$$A^{ab}(x; z) = A^{ab}(x) + 2\pi \mathbf{1}^{ab} z \tag{4.3}$$

with $z = z_1 + iz_2$ a complex coordinate on the dual torus $\hat{T}^2$. The mapping proceeds by considering the space of zero modes of the Dirac-Weyl equation defined by A. Let D and D^* denote the corresponding Dirac operator acting on right- and left-moving spinors, respectively. We will assume that D^* has no zero-modes. According to the index theorem, the number of zero-modes of D then equals the magnetic flux M

$$D\psi_i(x; z) = 0 \tag{4.4}$$

with $i = 1, \ldots, M$. We can choose an orthonormal basis of zero modes, satisfying

$$\int d^2x \, \overline{\psi}_i(x; z) \psi_j(x; z) = \delta_{ij} \tag{4.5}$$

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The dual $U(M)$ gauge field $\widehat{A}$ on $\widehat{T}^2$ is now defined as

$$\widehat{A}_{ij}(z) = \int d^2x\, \overline{\psi}_i^a(x;z)\widehat{\partial}\psi_{aj}(x;z) \tag{4.6}$$

with $\widehat{\partial} = \frac{\partial}{\partial \hat{z}}$. This definition is equivalent to the following formula for the dual covariant derivative $\widehat{D}$

$$\widehat{D}_\alpha \psi = (\mathbf{1} - \mathbf{P})\widehat{\partial}_\alpha \psi \tag{4.7}$$

where $\mathbf{P}$ denotes the projection on the space of zero modes ψ_i of the operator D. It can be shown that this dual gauge field has magnetic flux equal to N, and moreover that the mapping from A to $\widehat{A}$ is a true duality, in that it squares to the identity.

To make these properties more manifest, it is useful to obtain a somewhat more explicit form of the dual covariant derivative $\widehat{D}_\mu$. To this end, define the Green function

$$\Delta G(x,y) = \delta^{(2)}(x-y) \tag{4.8}$$

of the Laplacian $\Delta = DD^* + D^*D$, and introduce the notation

$$(\mathbf{G}\psi)(x;z) = \int d^2y\, G(x,y)\psi(y;z). \tag{4.9}$$

Then the projection operator $\mathbf{P}$ satisfies the relation

$$\mathbf{1} - \mathbf{P} = D^*\mathbf{G}D. \tag{4.10}$$

Inserting this identity into the definition (4.7), we find that

$$\widehat{D}^*\psi = D^*\mathbf{G}D\widehat{\partial}^*\psi = 0 \tag{4.11}$$

since $[D, \widehat{\partial}^*] = 0$. Hence the zero modes of the Dirac-Weyl operator on the dual torus are equal to the ones on the original torus, only with the opposite chirality. By an explicit construction [17] one can indeed verify that the dual $U(M)$ gauge field has magnetic flux equal to N, as predicted by the index theorem. Thus the Nahm transformation can be summarized in an elegant way by means of the two equations (4.4) and (4.11), which combined indeed completely specify the map from A to $\widehat{A}$. Note in particular that in this form, one of the 'magical' properties of the Nahm transformation has become manifest, namely that it is a map of order 2, i.e. it squares to the identity:

$$\widehat{\widehat{A}} = A, \tag{4.12}$$

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up to gauge equivalence. Notice further that the mapping is defined for arbitrary connections A.

When we translate the above construction back to our 3+1-dimensional setting, the resulting mapping indeed interchanges N and m_3 as advocated. In addition, since it is a mapping from T^2 to the dual torus $\widehat{T}^2$, the definition of the momentum flux in the direction of the T^2 gets interchanged with that of the electric flux along this direction, as indicated in (4.2). Naturally, the electric flux represents the conjugate momentum to the constant mode of the gauge fields A_i, and thus indeed defines the total momentum on the dual torus $\widehat{T}^2$. Further inspection also shows that the magnetic fluxes m_i get reflected, as predicted.

Finally, we should of course note that in the interpretation of the $U(N)$ SYM model as describing N D3 branes, this Nahm duality is nothing other than a double T-duality along the 1-2 directions of the three torus. [4] From this perspective, it in fact seems somewhat surprising that the Yang-Mills theory (via the Nahm transformation) still knows about this T-duality symmetry, in spite of the fact that it arises from string theory via the zero-slope limit.

References

[1] C. Montonen and D. Olive, "Magnetic monopoles as gauge particles," Phys. Lett. **72B** (1977) 117; P. Goddard, J. Nuyts and D. Olive, "Gauge theories and magnetic charge," Nucl. Phys. **B125** (1977) 1.

[2] H. Osborn, "Topological charges for $N = 4$ supersymmetric gauge theories and monopoles of spin 1," Phys. Lett. **83B** (1979) 321; A. Sen, Phys. Lett. **B329** (1994) 217, `hep-th/9402032`.

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[4]To recognize this interpretation of the mapping (4.1), see the translation code summarized in one of the previous footnotes. This interpretation of the Nahm transformation, as related to T-duality in string theory, was first suggested in [18].

[3] C. Vafa and E. Witten, "A strong coupling test of S-duality, " Nucl. Phys. **B431** (1994) 3, `hep-th/9408074`; L. Girardello, A. Giveon, M. Porrati, A. Zaffaroni, Nucl. Phys. **B448** (1995) 127, `hep-th/9502057`.

[4] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, "M theory as a Matrix Model: a Conjecture," Phys. Rev. **D55** (1997) 5112, `hep-th/9610043`.

[5] L. Susskind, "T-duality in M(atrix) Theory and S-duality in Field Theory," `hep-th/9611164`.

[6] W. Taylor, "D-brane Field Theory on Compact Spaces," Phys. Lett. **B394** (1997) 283, `hep-th/9611042`; O. Ganor, S. Ramgoolam and W. Taylor, "Branes, fluxes and duality in M(atrix) theory," Nucl. Phys. **B492** (1997) 191, `hep-th/9611202`.

[7] L. Motl, "Proposals on Nonperturbative Superstring Interactions," `hep-th/9701025`.

[8] T. Banks and N. Seiberg, "Strings from Matrices," `hep-th/9702187`.

[9] R. Dijkgraaf, E. Verlinde, and H. Verlinde, "Matrix String Theory," `hep-th/9703030`.

[10] T. Banks, N. Seiberg, and S. Shenker, "Branes from Matrices," Nucl. Phys **B490** (1997) 91, `hep-th/9612157`.

[11] L. Susskind, "Another Conjecture about M(atrix) theory," `hep-th/9704080`.

[12] G. 't Hooft, "A property of Electric and Magnetic Flux in Non-Abelian Gauge Theories," Nucl. Phys. **B153** (1979) 141.

[13] J. Harvey, G. Moore and A. Strominger, "Reducing S-duality to T-duality," Phys. Rev. **D52** (1995) 7161, `hep-th/9501022`.

[14] R. Dijkgraaf, G. Moore, E. Verlinde and H. Verlinde, "Elliptic genera of Symmetric Products and Second Quantized Srings," Commun. Math. Phys. **185** (1997) 197, `hep-th/9608096`.

[15] E. Witten, "Bound states of Strings and p-Branes," Nucl. Phys. **B460** (1996) 335, `hep-th/9510135`.

[16] P. J. Braam and P. van Baal, "Nahm's transformation for Instantons," Commun. Math. Phys. **122** (1989) 267; P. van Baal, "Instanton moduli for $T^3 \times \mathbf{R}$," Nucl. Phys. Proc. Suppl. 49 (1996) 238, `hep-th/9512223`.

[17] C. Hofman and E. Verlinde, unpublished.

[18] M. R. Douglas and G. Moore, "D-branes, Quivers, and A L E Instantons," `hep-th/9603167`.